Geometric Methods: Riemannian Geometry

RECAP: • Topology = study of shape 

• An n-dimensional manifold $\mathcal{M}$ is a topological space which "looks" <u>locally</u> likes $\mathbb{R}^n$.

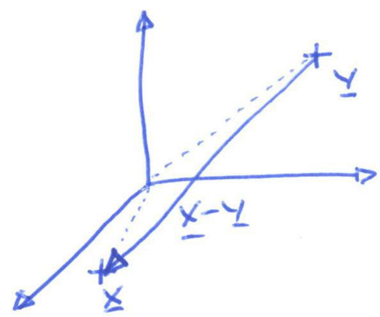
TODAY: • Geometry = study of length, angles, volume etc.

• Lengths useful e.g. distance to decision boundary, similarity.

In $\mathbb{R}^n$

$$d_{Euclid}(\underline{x}, \underline{y}) = \sqrt{\langle \underline{x}-\underline{y}, \underline{x}-\underline{y} \rangle_{Euclid}}$$

$$\langle \underline{u}, \underline{v} \rangle_{Euclid} := \underbrace{\sum_i u^i v^i}_{\text{distance depends on scalar product.}}$$



BILINEAR FORMS

Consider linear transformation $\underline{x} \longmapsto A\underline{\tilde{x}}$ so $x_i = \sum_j A_{ij} \tilde{x}_j$.

For scalar product to remain unchanged

$$\sum_i x_i y_i = \sum_i \left(\sum_j A_{ij} \tilde{x}_j\right)\left(\sum_k A_{ik} \tilde{y}_k\right)$$

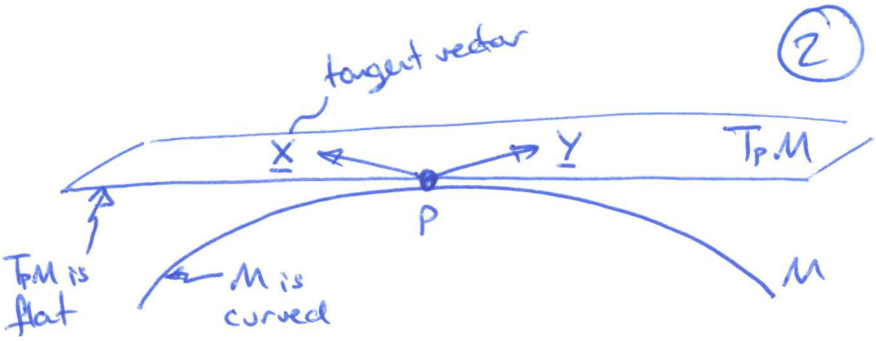
$$= \sum_{jk} \tilde{x}_j \tilde{y}_k \underbrace{\left(\sum_i A_{ij} A_{ik}\right)}_{G_{jk}}$$

$$= \sum_{jk} G_{jk} \tilde{x}_j \tilde{y}_k \quad \longleftarrow \text{most general form of scalar product}$$

symmetric positive def.

(M) 2

Previously we encountered tangent spaces T_pM "space of derivatives at $p \in M$"



Def A Riemannian Metric $g_p : T_pM \times T_pM \to \mathbb{R}_{\geq 0}$ is a symmetric, positive definite, bilinear form on tangent space at $p \in M$.

$$g_p(\underline{X}, \underline{Y}) = \sum_{ij} \underbrace{g_{ij}(p)}_{\text{"metric tensor"}} X^i Y^j$$

defined locally & everywhere + varies smoothly

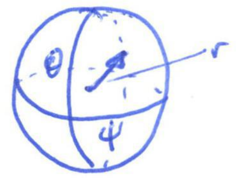
- Norm of a tangent vector $\|\underline{X}\|_{T_pM} = \sqrt{g_p(\underline{X}, \underline{X})}$

($\mathbb{R}^3$)

Example Infinitesimals

spherical: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$

cartesian: $ds^2 = dx^2 + dy^2 + dz^2$



Equate: $$[dr \;\; d\theta \;\; d\varphi] \underbrace{\begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{bmatrix}}_{g_{ij}(p) \text{ in spherical coords.}} \begin{bmatrix} dr \\ d\theta \\ d\varphi \end{bmatrix} = [dx \;\; dy \;\; dz] \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{g_{ij}(p) \text{ in cart.}} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

(M)

Def A Riemannian manifold (M, g) is a smooth manifold M equipped with a Riemannian metric g

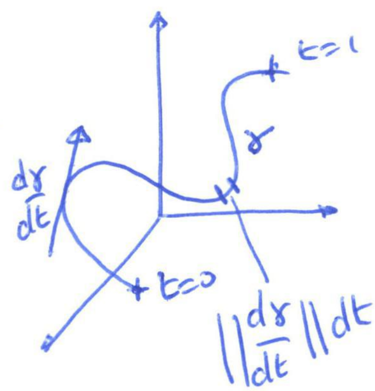
In ($\mathbb{R}^n$)

Given curve $\gamma : [0, 1] \to \mathbb{R}^n$

$$\underbrace{L(\gamma)}_{\text{length}} = \int_0^1 \underbrace{\left\| \frac{d\gamma}{dt} \right\|}_{\text{velocity}} \underbrace{dt}_{\text{time}} = \int_0^1 \Big(\sum_{ij} \delta_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \Big)^{1/2} dt$$

δ_{ij} = identity

only diff.



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In (M, g) $$\boxed{L(\gamma) = \int_0^1 \Big(\sum_{ij} g(\gamma(t)) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \Big)^{1/2} dt}$$

Def The (geodesic) distance between points $p, q \in M$ is the length of the shortest curve where $\gamma(0) = p$, $\gamma(1) = q$.

$$d(p,q) = \inf_{\substack{\gamma : \gamma(0)=p \\ \gamma(1)=q}} L(\gamma)$$

Geodesics generalize straight lines to manifolds.

Usually easier to minimize "energy" $E(\gamma) := \frac{1}{2}\int_0^1 \underbrace{\left(\sum_{ij} g_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\right)}_{\left\|\frac{d\gamma}{dt}\right\|^2_{T_pM}} dt$

Numerical solutions often only option. e.g. medical registration

Not a good ex

$$\text{minimize } \mathcal{E}(\phi) = \frac{1}{2}\underbrace{d_G(\mathrm{Id}, \phi)^2}_{\text{size of image warp.}} + \frac{1}{2}\int_0^1 |I \circ \phi^{-1}(y) - J(y)|^2 dy$$

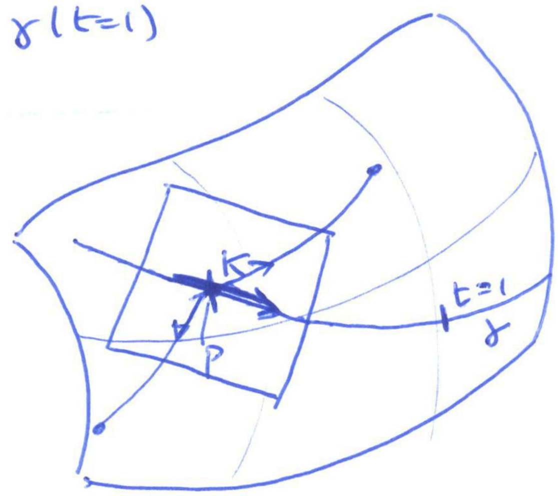
image warp — moving image — target image

Follow a geodesic γ through p in direction $\underline{K} \in T_pM$, $\gamma(0) = p$.

The exponential map $\exp_p : T_pM \rightarrow M$ is the point $\gamma(t=1)$ i.e.

$$\exp_p(\underline{K}) = \gamma(t=1)$$

- sometimes invertible
- not always one-to-one
- may not cover M

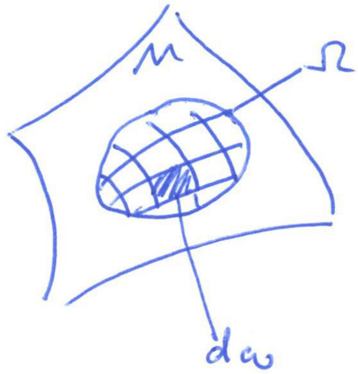


RIEMANN NORMAL COORDINATES

At p find basis change to make $g_{ij}(p) = \delta_{ij}$. $\exp_p$ does this magically!

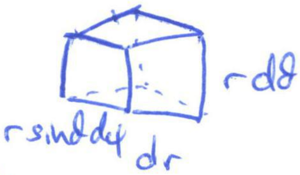
Volumes

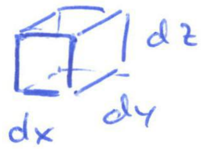
What is the volume of $\Omega \subset M$?



$$\text{volume} = \int_\Omega d\omega$$

Infinitessimal cube:





cart: $d\omega = dx\, dy\, dz$

sph: $d\omega = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$$G_{Euclid} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad G_{sph} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{bmatrix}$$

$$\det(G_{Euclid}) = 1 \qquad \det(G_{sph}) = r^4 \sin^2\theta$$

So maybe

$$\boxed{\text{volume} = \int_\Omega \underbrace{\sqrt{\det G}}\, dx_1\, dx_2, \ldots}$$

naturally accounts for changes of basis.
No more pesky Jacobians.